Singularity Analysis of CaPaMan: 
A Three-Degree of Freedom Spatial Parallel Manipulator

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Abstract
CaPaMan (Cassino Parallel Manipulator) is a three-degree of freedom parallel mechanism that has been designed and built at Laboratory of Robotics and Mechatronics in Cassino. In this paper a study of the configuration singularities of the CaPaMan manipulator is presented by considering two different methods: an algebraic formulation and a vector analysis. It will be shown that a formulation can give singularity related to the failure of the kinematic model at particular configurations of the manipulator. It will also be proved that this type of singularity can be avoided by a proper analysis of the problem.

1. Introduction
Parallel mechanical architectures were first introduced in tire testing by Gough and were later on used by Stewart as motion-simulators, [1]. However, for many applications a six-degree of freedom (dof) mechanism is not required and three-dof manipulators may be used. For instance, three-dof mechanisms have been presented for telescope applications [2]; flight simulation [3]; beam aiming applications [4].

Although three-dof spatial parallel manipulators present several advantages, such as a reduction of the total cost of the device including manufacturing and operations, singularity analysis can be complicate since position and orientation of the mobile platform with respect to the fixed frame are coupled. This means that the configuration can be completely described by using only three of the six variables (i.e. the Cartesian coordinates of the operation point and three Euler angles).

The physical meaning of a singularity in Kinematics refers to those configurations in which the number of dof of the mechanism changes instaneously. Algebraically, a singularity is related to a rank deficiency of the associated Jacobian matrices while, geometrically, it is observed whenever the manipulator gains some additional, uncontrollable degrees of freedom, or loses some dof, [5].

The concept of singularity has been extensively studied and several classification methods have been defined. Gosselin and Angeles suggested in [6] a classification of singularities for parallel manipulators into three main groups. The first type of singularity occurs when the manipulator reaches internal or external boundaries of its workspace and the output link loses one or more dof. Second type of singularity is related to those configurations in which the output link is locally movable even if all the actuated joints are locked. Third type is related to linkage parameters and occurs when both first and second type of singularities is involved.

Later, Ma and Angeles [7] introduced another classification for singularities, namely configuration singularities, architecture singularities and formulation singularities. First type of singularity is related to particular configurations of the manipulator. Architecture singularities are caused by certain architectures; they do not depend from the specific configuration of the manipulator. Formulation singularities are due to the adopted analysis and they can be avoided simply by changing formulation method.

In this paper two different methods are presented for the determination of the configuration singularities of the CaPaMan manipulator: an algebraic formulation and a vector analysis. In particular, it will be shown that a formulation singularity caused by the failure of the kinematic model at particular configurations of the manipulator may be encountered.

CaPaMan (Cassino Parallel Manipulator) is a three-degree-of-freedom parallel mechanism that has been designed and built at Laboratory of Robotics and Mechatronics in Cassino. Mechanical and kinematical characteristics of the manipulator have been extensively studied in [8; 9].

2. Mechanism Architecture
A schematic representation of the CaPaMan manipulator is shown in Fig.1 where the fixed plate is labeled as FP and the moving platform is MP. Indeed MP is connected to FP through three identical leg mechanisms and is driven by the corresponding articulation points H₁, H₂, H₃. A built prototype is shown in Fig. 2.

Each leg mechanism is one-dof and is composed by three parts: an articulated parallelogram AP, a prismatic joint SJ and a connecting bar CB. AP’s coupler carries the SJ and CB transmits the motion from AP to MP through
Fig. 1 Kinematic architecture and design parameters of CaPaMan.

Fig. 2. A built prototype of CaPaMan (Cassino Parallel Manipulator) at Laboratory of Robotics and Mechatronics in Cassino.

SJ; CB is connected to the MP by a spherical joint BJ, which is installed on MP. CB may translate along the prismatic guide of SJ keeping its vertical posture and BJ allows MP to rotate in the space. Each plane, which contains AP, is rotated of \( \pi/3 \) with respect to the neighbor one.

Particularly, links of the \( k \)-th leg of the mechanism, are identified through: \( a_k \), which is the length of the frame link; \( b_k \), which is the length of the input crank; \( c_k \), which is the length of the coupler link; \( d_k \), which is the length of the follower crank; \( h_k \), which is the length of the connecting bar. The kinematic variables are: \( \alpha_k \), which is the input crank angle; \( \alpha_k \), which is the stroke of the prismatic joint. Finally, the size of MP and FP are given by \( r_p \) and \( r_f \), respectively. \( \Omega \) is the center point of FP, \( H_k \) is the center point of the \( k \)-th BJ and \( O_k \) is the middle point of the frame link \( a_k \), Fig.1.

The motion of MP with respect to FP can be described by considering a world frame O-XYZ, which is fixed to FP, and a moving frame H-XpYpZp, which is fixed to MP and a AP plane reference frame \( O_k -X_k Y_k Z_k \). Z-axis is chosen orthogonal to the FP plane, X-axis is coincident with the \( O O_1 \) line and Y-axis forms Cartesian frame. Zp is orthogonal to the MP plane, Xp axis is coincident with the \( HH_1 \) line and \( Y_1 \) axis gives a Cartesian frame.

3. Velocity Analysis

Velocity equations represent the linear mapping between joint and Cartesian velocities. They are important because they characterize the kinematic accuracy of the mechanism and also allow determining its singularities.

The differential kinematic relation for parallel manipulators can be expressed in the form

\[
J\dot{\theta} = K\dot{t}
\]

where \( J \) and \( K \) are two Jacobian matrices of the manipulator. Moreover \( \dot{\theta} \) is the vector of joint rates, \( \dot{t} = [\alpha_1, \alpha_2, \alpha_3]^T \), and \( \dot{t} \) is the twist array, which assumes different forms, depending on the nature of the task space, which could be planar, spherical or spatial. \( T \) stands for the transpose operator. In the foregoing form \( \dot{t} \) is a six-dimensional array for spatial tasks, \( \dot{t} = [v_H^T, \omega]^T \), where \( \omega \) is the three-dimensional angular velocity vector of the moving platform and \( v_H \) is the three-dimensional velocity of the operation point \( H \) of the moving platform.

Two different methods are discussed for the determination of the velocity equations: an algebraic formulation and a vector analysis. The first method consists in differentiating the relations obtained from the Direct Kinematic analysis. The latter refers to closed-loop equations for the manipulator written as function of linear and angular velocities of its links.

3.1 Algebraic Formulation

The algebraic formulation for velocity equations consists in directly differentiating the equations obtained from the kinematic analysis of the manipulator.

In order to specify position and orientation of the mobile platform with respect to the fixed frame it is necessary to consider six variables. These ones could be chosen as the Cartesian coordinates of the operation point \( H \) and the Euler angles \( \varphi, \theta \) and \( \psi \), as shown in Fig.1.

Thus, the differential kinematic relation of Eq.(1) can be reformulated in the form

\[
J\dot{\theta} = K A \dot{t}_E
\]
where \( \mathbf{J}_t = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} & \dot{\phi} & \dot{\psi} & \dot{\theta} \end{bmatrix}^T \) and \( \mathbf{A} \) is a \([6 \times 6]\) transformation matrix which assumes the form

\[
\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(3)

\( I \) is the Identity matrix and 0 stands for a \([3 \times 3]\) zero matrix. \( \mathbf{R}' \) gives the relationship between the angular velocity of the movable platform \( \omega \) and the time derivatives of the Euler angles. According with the used convention \( \mathbf{R}' \) assumes the form

\[
\mathbf{R}' = \begin{bmatrix} s^2 \phi s \theta & (c \phi + c \psi s \theta) c \phi & 0 \\
-s^2 \phi c \theta & (s \phi - c \psi c \theta) c \phi & 0 \\
0 & s \phi & 1 \\
\end{bmatrix}
\]  

(4)

where \( c \) stands for cosine and \( s \) for sine.

However, by using the algebraic formulation we have introduced other singularities, which can be referred as formulation singularities. They arise when the matrix \( \mathbf{R}' \) becomes singular, in other words if \( \phi \) is equal to 0, 90 or 180 deg. This type of singularity is due to the analysis and can be avoided by a proper formulation of the problem. Since the mechanism has three-dof, only three of the six variables can be specified as function of the input crank angles \( \alpha_k, (k=1,2,3) \), for describing the configuration of CaPaMan. These coordinates can be chosen as two rotations about two perpendicular axes intersecting at the mobile platform center, \( \phi \) and \( \psi \), and a vertical translation, \( z \): we call them independent. The other coordinates \( x, y \) and \( \theta \), can be specified by using a proper formulation of the kinematic analysis or determined with constraint equations associated to the mechanism architecture: these are defined dependent.

Thus, Eq.(2) can be rewritten as

\[
\mathbf{J}\dot{\theta} = \mathbf{K}_e \mathbf{A}_R \mathbf{A} \mathbf{R}_t \mathbf{i}_r
\]  

(5)

where \( \mathbf{A}_R \) expresses the relationship between the independent, \( \mathbf{i}_r = [\phi \ \psi \ \dot{z}]^T \), and dependent coordinates.

As previously determined in [8], it is possible to specify independent coordinates as function of the input crank angles only as

\[
\begin{align*}
\dot{z} &= \frac{z_1 + z_2 + z_3}{3} \\
\phi &= \cos^{-1} \left( \frac{2}{3} \sqrt{E} \right) \\
\psi &= \tan^{-1} \left( \sqrt{\frac{D-F}{D+F}} \right)
\end{align*}
\]

(6)

with

\[
\begin{align*}
E &= z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_1 z_3 \\
D &= 2 z_2 - z_1 - z_3 \\
F &= 2 z_3 - z_1 - z_2
\end{align*}
\]  

(7)

when for \( k = 1,2,3 \), one consider

\[
z_k = b_k \sin \alpha_k
\]  

(8)

Differentiating Eqs.(6) with respect to time yields

\[
\dot{\phi} \sqrt{E} \left[ \begin{array}{c} \dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 
\end{array} \right] = \dot{z}_1 (D-F) + \dot{z}_2 (D+2F) - \dot{z}_3 (D+2F)
\]

(9)

\[
\psi \frac{6E}{\sqrt{3}} = \dot{z}_1 (D+F) + \dot{z}_2 D - \dot{z}_3 F
\]

with

\[
\dot{z}_k = b_k \cos \alpha_k \alpha_k
\]  

(10)

From Eq.(5) the Jacobian matrices associated with the CaPaMan manipulator can be written as

\[
\mathbf{K}_e = \begin{bmatrix} 6E & 0 & 0 \\
0 & \sqrt{E} \left[ \begin{array}{c} \dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 
\end{array} \right] & 0 \\
0 & 0 & 3 
\end{bmatrix}
\]  

(11)

\[
\mathbf{J} = \begin{bmatrix} (D-F) b_1 c \alpha_1 & (D+2F) b_2 c \alpha_2 & -(2D+F) b_3 c \alpha_3 \\
(D+F) b_1 c \alpha_1 & -Db_2 c \alpha_2 & -F b_3 c \alpha_3 \\
b_1 c \alpha_1 & b_2 c \alpha_2 & b_3 c \alpha_3 
\end{bmatrix}
\]

(12)

where \( \mathbf{K}_e \) expresses \( \mathbf{K} \mathbf{A} \mathbf{A}_R \).

Equations (11) and (12) represent the Jacobian matrices associated to CaPaMan manipulator, which have been derived by using an algebraic formulation.

### 3.2 Vector Analysis

The second approach for deriving the velocity equations consists of writing closed-loop velocity equations as function of linear and angular velocities of the links of the mechanism, [10]. This analysis leads to an invariant form of the Jacobian matrices. The velocity vector of an articulation point \( \mathbf{H}_k \) is formulated from two different loop-closure circuits. Each one consists of the FP, MP and links of the leg, as shown in Fig.3. The unactuated joint rates in each leg are passive variables and they are eliminated by perform a dot product of the velocity vector-loop equation with an appropriate vector, which is orthogonal to all vectors of unactuated joint rates. Thus the resulting equations can be assembled in
the Jacobian matrices. It is assumed that, unless otherwise indicated, all vectors and matrices are represented in the world frame.

By considering Fig.3, for each k-th leg a closed-loop equation can be written as

$$k H H k ' R r p p + =$$

for $k=1,2,3$ (13)

$r_k'$ is the vector connecting $HH_k$ and it is defined in the moving frame $H-X P Y P Z P$ and $R$ is the rotation matrix which describes the orientation between the moving and fixed frame. By considering Fig.3, Eq.(13) can be also written as

$$k k k k k H h s l u p + + + =$$

(14)

Differentiating Eqs.(13) and (14) with respect to time yields

$$v_H + \Omega r_k = \omega_k \times l_k + s_k$$

(15)

where $\Omega = \dot{R} R^T$; $\omega_k$ is the angular velocity of the k-th leg with respect to the AP plane reference frame; $s_k$ is the linear velocity of the prismatic joint.

To eliminate the joint rates $s_k$ in Eq.(15), which are passive variables, we dot-multiply both sides by $h_k$, which is a constant vector for all the three legs. This gives

$$h_k \cdot v_H + h_k \cdot \Omega r_k = h_k \cdot \omega_k \times l_k$$

(16)

where “.” is the dot product and “×” is the cross product between vectors. If one considers

$$h_k \cdot \omega_k \times l_k = (l_k \times h_k)^T \omega_k$$

(17)

and

$$h_k \cdot \Omega r_k = (r_k \times h_k)^T \omega$$

(18)

Substituting Eqs.(17) and (18) in Eq.(16) the closed-loop velocity equations associated to the leg mechanisms can be written as

$$l_k \times h_k \cdot \omega_k = h_k \cdot v_H + r_k \times h_k \cdot \omega$$

(19)

Writing Eq.(19) for each leg, the velocity equations associated with the vector analysis can be expressed as

$$J_v \left[ \omega_1^T \ \omega_2^T \ \omega_3^T \right]^T = K \mathbf{t}$$

(20)

Thus, the Jacobian matrices associated to the CaPaMan manipulator can be written as

$$J_v = \begin{bmatrix} (l_1 \times h_1)^T & 0 & 0 \\ 0 & (l_2 \times h_2)^T & 0 \\ 0 & 0 & (l_3 \times h_3)^T \end{bmatrix}$$

(21)

and

$$K = \begin{bmatrix} (r_1 \times h_1)^T \\ (r_2 \times h_2)^T \\ (r_3 \times h_3)^T \end{bmatrix}$$

(22)

where $\mathbf{0}$ denotes $[1x3]$ vector. The obtained Jacobian matrices are $J_v$, which is a $[3x9]$, and $K$, which is a $[3x6]$.

4. Singularity Analysis

In parallel manipulators, singularities arise whenever $J$, $K$ or both, becomes singular. Thus, a distinction can be made among three types of singularities, [6], which have different kinematic interpretations by considering Eq.(1), namely:

1) The first type of singularity occurs when $J$ becomes singular but $K$ is invertible, in other words

$$\text{det}(J)=0 \quad \text{and} \quad \text{det}(K)\neq 0$$

(23)

2) The second type of singularity occurs only in closed kinematic chains and arises when $K$ becomes singular but $J$ is invertible, i.e.

$$\text{det}(J)\neq 0 \quad \text{and} \quad \text{det}(K)=0$$

(24)

3) The third type of singularity occurs when $J$ and $K$ are simultaneously singular, while none of the rows of $K$ vanishes. Under a singularity of this type, configurations arise for which the movable plate can undergo finite motions even if the actuators are locked or, equivalently, it cannot resist to forces or
moments into one or more directions over a finite portion of the workspace, even if all actuators are locked. A finite motion of the actuators gives no motion of the mobile plate and some Cartesian velocity vectors cannot be obtained.

5. Singularity Analysis for CaPaMan

We have applied the above-mentioned formulation to CaPaMan for a specific study of its singularities.

The first type of singularity occurs when the determinant of J vanishes. By considering the vector analysis the Eq.(21) yields the condition, for i = 1 or 2 or 3,

\[(h_k \times h_k) = 0\]  

(25)

This type of singularity arises whenever any input crank angle of the articulated parallelogram is aligned with the connecting bar \(h_k\). In other words whenever any leg is in a fully extended configuration the manipulator loses 1, 2 or 3 degrees of freedom, depending on the number of legs which are in that condition.

By considering the algebraic formulation, from Eq.(12) one obtains the condition

\[\text{det}(J) = c_1 c_2 c_3 \left[ b_1^2 s \alpha_1^2 + b_2^2 s \alpha_2^2 + b_3^2 s \alpha_3^2 - b_1 b_2 s \alpha_1 \alpha_2 - b_1 b_3 s \alpha_1 \alpha_3 - b_2 b_3 s \alpha_2 \alpha_3 \right]\]  

(26)

By considering Eqs.(7) and (8), Eq.(26) can be written as

\[\text{det}(J) = E c_1 c_2 c_3\]  

(27)

From Eq.(27) it is possible to note that whenever any input crank angle is equal to 90 deg the manipulator is on the first type of singularity. This result is in agreement with the one obtained with vector analysis.

In order to investigate the condition \(E = 0\), if \(b_i = b_j\), for \(i \neq j\), \(i,j=1,2,3\), and assuming \(u = c \alpha_1\); \(v = c \alpha_2\) and \(w = c \alpha_3\), E coefficient in Eq.(27) can be written as

\[E = u^2 + v^2 + w^2 - uv - uw - vw\]  

(28)

that can be also expressed as

\[E = \frac{1}{2} \left[ (u-v)^2 + (u-w)^2 + (v-w)^2 \right]\]  

(29)

Equation (29) is equal to zero if \(u = v = w\), whenever three input crank angles have the same value.

Considering Eq.(6) the condition \(E = 0\) is satisfied if \(\varphi\) is 90 deg for the specific adopted formulation.

This is a formulation singularity due to the fact that for this case effects of the first and third rotations \(\varphi\) and \(\theta\) cannot be distinguished. This type of singularity has not a physical meaning and it can be avoided if the vector analysis is considered.

The second type of singularity arises when matrix \(K\) in Eq.(22) is rank deficient by considering the vector formulation. The rank of a matrix equals the maximal number of independent rows or columns, so the condition is verified by imposing the linear dependence of the columns or rows of \(K\). From Eq.(22), \(K\) becomes singular if its rank is equal to 2 or 1. In particular \((r_k \times h_k)^T\) represents a vector which lays in the MP plane, so all three vectors, for \(k = 1,2,3\) are linearly dependent because all lay in that plane, but \(K\) is not rank deficient. Even if only one of the three vectors has zero components the matrix \(K\) has still rank equal to three. If two or all the three vectors have zero components \(K\) becomes singular. This condition implies that the platform is aligned with the connecting bars.

By considering the algebraic formulation Eq.(11) gives the conditions

\[E = 0\]  

(30)

or

\[\frac{2\sqrt{E}}{3r_p} = 1\]  

(31)

Equation (30) represents a formulation singularity and it has been previously analyzed. By considering Eq.(6), Eq.(31) is verified whenever \(\varphi\) is equal to 0 or 180 deg, when the mobile platform is in its vertical posture. This result is in agreement with the one obtained with vector analysis.

In order to analyze the configuration of CaPaMan in its second type of singularity one can study the condition expressed by Eq.(31), together with Eq.(28). This condition represents a cylinder with elliptical cross section. The cross section plane is perpendicular to the unit vector \(n=[1,1,1]^T\). An intersection of the elliptic cylinder with a plane \(x+y+z = 0\) is its elliptic directrix with center in point \(D(0, 0)\), semimajor axis \(p = \pm \frac{m}{3}\); semiminor axis \(q = \pm \frac{m^2}{\sqrt{3}}\) with \(m = \frac{3r_p}{2}\).

The elliptic cylinder divides the space into two regions free from singularities: the region inside and outside the cylinder.

The condition expressed by Eq.(31) does not lead to practical solutions, in fact in order to be verified \(r_p\) must have a value less than 1. This is due to cosine function limitation for \(u, v\) and \(w\).

6. Conclusions

CaPaMan is a 3-dof spatial parallel manipulator that has been designed and built at Laboratory of Robotics and Mechatronics in Cassino.

In this paper a singularity analysis of CaPaMan has been presented. In particular, two different approaches
have been used: an algebraic formulation and a vector analysis. Singularities have been determined in analytic form. It has been shown a formulation singularity related to the algebraic formulation: it can be avoided by considering the vector analysis. In fact, using this method the Jacobian matrices can be derived in an invariant form.

Configuration singularities for CaPaMan have been analyzed: whenever any leg is in a fully extended configuration the manipulator is in the first type of singularities; if the mobile platform is in its vertical posture CaPaMan is in the second type of singularities.

References

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